MATHEMATICAL MODELS OF TRANSPIRATION COOLING

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Possible identification of two mathematical models of transpiration cooling is analyzed. Processes with heat supply in parallel with or normal to the direction of the injection of a coolant are considered.

As is known, two models are used for calculating thermal fields in a porous plate in the case when heat supply is in parallel with the direction of injection of a coolant. The first of them can be called a model of two continua: in this model, the coolant and a solid phase are considered as different continua enclosed in the same geometrical volume determined by the plate. The model assumes, generally speaking, a certain difference in temperatures at any point in the volume of the continua. The bulk velocity of heat exchange between the solid phase and the coolant is described by the expression $\alpha_V(T-\tau)$. The thermal fields are determined from the solution of the following system of equations (it is assumed that heat conduction of the coolant can be neglected):

$$d^2T/dx^2 = \pi_x d\tau/dx,$$
$$d\tau/dx = \sigma(T-\tau). \quad (1)$$

The boundary conditions at the inlet and the outlet of the plate for the general case are written as follows:

$$-dT_a/dx = \pi_x\sigma_a(\tau_- - T_a) + q_a^x, \quad (2)$$

$$-dT_b/dx = \pi_x\sigma_b(T_b - \tau_b) + \beta_b(T_b - \Theta_b) + q_b^x. \quad (3)$$

In Eq. (3), we use separate terms for heat flows from the wall to the coolant and from the wall to the ambient medium, in which the coolant is injected. As a ground for such separation might serve the fact that heat exchange between the wall and the coolant is always the case (of course, when the wall is supplied with the heat energy), while heat exchange between the wall and the ambient medium may also be nonexistant.

We note that the choice of the thermal head is not fundamental for defining the criteria σ_a, σ_b. In Eqs. (2), (3), its maximal value is used both at the inlet and at the outlet.

Proceeding from the energy balance, one can readily obtain

$$\tau_a = \sigma_a T_a + (1-\sigma_a)\tau_-, \quad (4)$$
$$\tau_+ = \sigma_b T_b + (1-\sigma_b)\tau_b. \quad (5)$$

Formulas (1)-(5) represent a mathematical description of the two-continuum model of transpiration cooling.

The one-continuum model is used alongside with the two-continuum model. The former assumes the identity of the temperaures of the solid phase and the coolant throughout the entire volume of the wall, and the heat sink toward the coolant is described by the expression $\dot{m}c_p dT/dx$. The equation defining the thermal field is of the form

$$d^2T/dx^2 = \pi_x dT/dx. \quad (6)$$

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The boundary condition at the inlet in the porous wall deserves special mention. Using the identity $T \equiv \tau$, accepted for the volume of the wall, toward the surface of the outlet, we obtain $T_a = \tau_a$. This equality expresses the fundamental for the one-continuum model conjecture, according to which the coolant "must" be heated up to the temperature of the wall by the moment of penetrating it. (To be more exact, it is, of course, suggested that heating "must" occur in a negligibly thin layer of a porous metal.) Heat flow required for such heating is, apparently, equal $\pi_x(\tau_- - T_a)$. In correspondence with this, by analogy with (2), we write

$$-dT_a/dx = \pi_x(\tau_- - T_a) + q_a^x. \tag{7}$$

It should be emphasized that Eq. (7) is the only substantiated boundary condition on the surface of the inlet for the one-continuum model.

On the surface of the outlet, thermal flow from the porous wall to the coolant is equal to zero since their temperatures are equal, therefore

$$-dT_b/dx = \beta_b(T_b - \Theta_b) + q_b^x. \tag{8}$$

Equations (6)-(8) give a mathematical description of a one-continuum model of transpiration cooling.

Comparing the given models, an inference can be made that the two-continuum model is physically a more convincing one, however, requiring more information and being more complicated in a mathematical sense. As for the one-continuum model, it is less convincing physically (it is unlikely necessary to prove that the identity between the temperature of the porous material and the coolant cannot exist). However, this model is simpler in a mathematical sense and does not require the knowledge of such scarcely studied values as the coefficients of heat transfer in the porous material and at its inlet and outlet.

It might be well to point out the works using both the two-continuum [1-6] and the one-continuum [7-9] models.

It is natural to pose a question on how much the solution of system (1) differs from that of Eq. (6). Under conditions of sufficient closeness of these solutions, we will assume that it is possible to identify the two models, otherwise, it is impermissible.

Apparently, the closeness of solutions (1), (6) depends on the degree of closeness of the temperatures of the solid phase and of the coolant along the thickness of the porous wall. As can be readily seen from the second equation of system (1), the difference $\tau - \tau$ determines two values: the criterion σ and the derivative $d\tau/dx$. Therefore, considering the limitation of the derivative, the problem of possible identification of the two models from a mathematical point of view reduces to the determination of the limiting criterion σ (designated as σ_∞) starting from which the solution of system (1) differs sufficiently little from the solution of Eq. (6). Since the derivative $d\tau/dx$ is a function of the criteria σ, π_x, σ_a, σ_b and of the type of the boundary conditions, it is completely clear that in the general case σ_∞ is a function of the criteria π_x, σ_a, σ_b and the type of the boundary conditions. Below we define the form of this function.

We introduce the parameter η, by which we will evaluate the closeness of solutions (1) and (6): $\eta = \max(|\eta_T|, |\eta_\tau|)$. The value of η_T determines the difference in the maximal temperatures of the wall, calculated with the help of the one- and two-continuum models: $\eta_T = (T^{(6)}_{max} - T^{(1)}_{max})/T^{(1)}_{max}$. The superscript shows that the temperature is obtained from solution (1) or (6). The value η_τ determines the difference in the heating temperatures of the coolant, calculated with the help of the models under consideration: $\eta_\tau = (T^{(6)}_b - \tau_+)/(\tau_+ - \tau_-)$.

Thus, the use of the parameter η allows one to evaluate the closeness of solutions (1) and (6) from the two factors that are of greatest practical importance: maximum temperature of the wall and the heating temperature of the coolant.

The calculation process was as follows. In Eqs. (2), (3), (7), (8) one of the terms describing the ambient supply of heat energy to the wall was alternately nonzero; i.e., in other words, the type of the boundary conditions was changed. For each case, the criterion σ_a was varied from zero to one. The criterion σ_b in (3) was always thought to be equal to σ_a. (Such an assumption can be "justified" by the fact that for the problem in question,

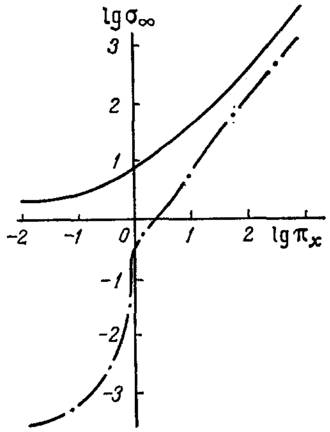

Fig. 1. Dependence of σ_∞ on π_x and σ_a for transpiration cooling with heat supply parallel to the direction of injection: solid line, $\sigma_a = 0$; dot-dash line, $\sigma_a = 0.5$. For $\sigma_a = 1$ $\sigma_\infty < 10^{-3}$.

solution (1) was of interest to us for those σ for which the difference $T-\tau$ along the thickness of the wall is sufficiently small. In this case, the difference $T_b-\tau_b$ is even smaller, and, consequently, the corresponding heat flow is small. Consequently, in the case under consideration, the criterion σ_b did not practically affect the structure of the solution of system (1).)

Thus, by noting π_x for each type of the boundary conditions and for a certain σ_a, we selected a value of the criterion σ from the interval 10^{-3}-10^3 so that η was between 0.15-0.20. Such a value of σ was taken as σ_∞ for the case under consideration. Then by varying the criteria π_x, σ_a and the type of the boundary conditions, we obtained a number of values of σ_∞, which allowed us to determine the form of the studied function.

We note that system (1) has been solved numerically, with a certain error that is characteristic of the method. That is why the results give below, in particular, the form of the function σ_∞, are approximate in nature.

The analysis of the results of calculations allowed us to find that the value σ_∞ did not depend practically on the type of the boundary conditions and, thus, was a function of the two arguments only, π_x, σ_a (see Fig. 1).

As is seen from Fig. 1, the complete set of values of the criterion σ can be divided into two regions. In the first region, situated above the solid line in Fig. 1, solutions (1) and (6) independently of the criteria π_x and σ_a differ little (in the sense of inequality $\eta < 0.2$) from each other, i.e., in this region, the two models can be identified unquestionably. In the second region, situated below the solid line, the difference between solutions (1) and (2) depends already on the criteria π_x and σ_a. As this takes place, the difference is small when the inequality $\sigma \geq \sigma_\infty$ holds, otherwise, $\eta > 0.2$ and the difference can be significant. Thus, in the second region, the identification of two models is possible observing a certain limitation on the criteria σ, π_x, and σ_a only.

The value of σ_∞ depends strongly on π_x, σ_a, being extremely small for $\sigma_a = 1$ (at any rate, less than 10^{-3}), and of order of π_x for $\sigma_a = 0$ (for $\pi_x > 1$).

We consider the models of transpiration cooling with heat supply normal to injection and with bulk heat evolution.

The mathematical description of the two-continuum model is as follows:

$$\varkappa\partial^2T/\partial x^2+\partial^2T/\partial y^2=\pi\partial\tau/\partial x-q,\quad d\tau/\partial x=\sigma(T-\tau). \tag{9}$$

The boundary condition on the surface of the coolant inlet to the plate is of the form

$$-\partial T_a/\partial x=\pi_x\sigma_a(\tau_--T_a). \tag{10}$$

The boundary condition on the surface of the outlet is

$$-\partial T_b/\partial x=\pi_x\sigma_b(T_b-\tau_b). \tag{11}$$

On the lower and upper sides of the plate, the boundary conditions in the general case are written as

$$-\partial T_n/\partial y=\beta_n(\Theta_n-T_n)+q_n^y, \tag{12}$$

$$-\partial T_v/\partial y=\beta_v(T_v-\Theta_v)+q_v^y. \tag{13}$$

In order to obtain the complete set of boundary conditions, we have to add Eq. (4) to conditions (10)-(13). The temperature of the coolant far from the surface of the outlet can be determined from Eq. (5).

The mathematical description of the one-continuum model is

$$\varkappa\partial^2T/\partial x^2+\partial^2T/\partial y^2=\pi\partial T/\partial x-q. \tag{14}$$

To derive the boundary condition at the coolane inlet at a porous wall, we will use the principal conjecture of the one-continuum model, according to which the coolant "must" reach the temperature of the wall in a negligibly thin layer of the porous metal. In this case, by analogy with (7) we can readily obtain

$$-\partial T_a/\partial x=\pi_x(\tau_--T_a). \tag{15}$$

The boundary condition at the coolant outlet in the absence of heat exchange with the ambient medium and in the absence of heat flows with consideration of Eq. (8) is of the form

$$\partial T_b/\partial x=0. \tag{16}$$

The boundary conditions on the upper and lower sides of the plate for the one-continuum model are formulated by analogy with (12) and (13).

As is shown above, from the mathematical point of view, the problem of possible identification of the two-continuum model with the one-continuum model reduces to the determination of a limiting value of the criterion $\sigma = \sigma_\infty$, starting from which the solution of system (9) differs sufficiently little from solution (14). Apparently, in the general case σ_∞ is a function of the criteria π, κ, σ_a, σ_k and of the type of the boundary conditions on the lower and upper sides of the plate. The form of this functional dependence is investigated below.

As a preliminary measure, we note a few qualitative results concerning the function σ_∞. We consider the problem with bulk heat evolution. In this case, the first equation of system (9) takes the form

$$\varkappa d^2T/dx^2=\pi d\tau/dx-q. \tag{17}$$

Letting $\kappa = 0$ in this equation (physically, this is possible, for example, for $\lambda_x = 0$) and using the second of Eqs. (9), we obtain

$$T-\tau=(q/\pi)/\sigma.$$

The expression in parentheses on the right side, as is readily seen, determines the heating of the coolant. We choose the scale for the temperature T from the condition that the heating is equal to unity (we recall that T enters the denominator of q). Then we obtain

$$T-\tau=1/\sigma. \tag{18}$$

Thus for problems where the conductive heat flow in the direction of the injection can be ignored, the ratio $1/\sigma$ estimates the order of smallness of $T-\tau$. As calculations show, for problems with heat supply normal to injection and with bulk heat evolution, the difference from zero of the term $\kappa\partial^2T/\partial x^2$ in system (9) and Eq. (14) plays an insignificant role for the problem in question, and, thus, Eq. (18) reflects rather precisely the real ratio of the temperatures. Based on this, it is natural to expect that the function σ_∞ depends weakly on its arguments and is of the order of ten. Calculations support this conclusion.

We pass now to the description of the calculation process. Heat exchange with the ambient medium was specified with the help of Eq. (12), in which either β_n or q_n^y were assumed to be

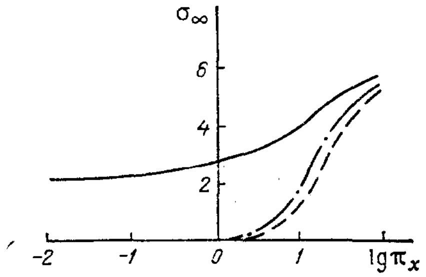

Fig. 1. Dependence of σ_∞ on π_x and σ_a for transpiration cooling with bulk heat evolution: solid line, $\sigma_a = 0$; dot-dash line, $\sigma_a = 0.5$, dashed line, $\sigma_a = 1$.

different from zero. The upper side of the plate was assumed to be heat-insulated; for the problem with bulk heat evolution, the upper and the lower sides of the plate were assumed to be heat-isolated. We varied the criteria κ and σ_a (in the range 0-1), π (in the range 10^{-2}-10^2), σ_b was always assumed to be equal to σ_a (see above for a remark with regard to this). The closeness of the solutions of system (9) and Eq. (14) is evaluated from two parameters: η_T and η_τ, similar to those introduced above. The limiting value of the criterion $\sigma - \sigma_\infty$ was selected from the requirement that $\eta = \max(|\eta_T|, |\eta_\tau|)$ was between 0.15 and 0.20.

System (9) and Eq. (14) were solved numerically. In connection with this the results stated below have a certain "error" in the framework of the error of the numerical method.

As a result of calculation, it has been found that for problems in which heat energy supply to the plate is determined by Eq. (12), the form of the limiting function does not depend practically on the nature of heat exchange with the external source and variations in the criteria κ, π, and σ_a affect the value of σ_∞ rather weakly. Thus, the limiting function for problems with heat supply normal to injection reduces simply to a constant. Its value is determined by the following approximate equality $\sigma_\infty \approx 6$.

The form of the limiting function is somewhat more complicated for the problem with bulk heat evolution. (First of all, we note that by dividing the first of Eqs. (9) by κ and by making an appropriate selection for the scale of the temperature T, it is possible to exclude κ from a number of the arguments σ_∞.) As calculations show, σ_∞ in this case depends on σ_a and π_x (see Fig. 2).

In correspondence with Fig. 2, the entire set of values of the criterion σ for the problem with bulk heat evolution can be divided in three regions. In the first region, situated above the solid line in Fig. 2, the solution of system (9) and that of Eq. (14) differ little regardless of the criteria π_x and σ_a, and, consequently, the two models can certainly be identified. In the second region, bounded by the solid and dashed lines, identification is possible, if the inequality $\sigma \geq \sigma_\infty$ holds. Otherwise, the difference between solutions (9) and (14), generally speaking, can be significant. In the third region, situated below the dashed line, identification is inadmissible, irrespective of the criteria σ_x and π_a.

The value of the limiting function for the problem with bulk heat evolution is small, and in the region of small π_x it depends significantly on the criterion α_a, with increase of π_x the effect of the σ_a weakens, and the limiting function tends, apparently, to its asymptote, approximately equal to 6.

As for the problems with heat supply normal to injection, for them, in correspondence with the aforementioned, the entire set of values of the criterion σ can be divided into two regions. In the first, where $\sigma \geq 6$, identification of the two models is admissible; in the second, where $\sigma < 6$, the difference between solutions (9) and (14), generally speaking, can be significant and, therefore, identification is inadmissible.

A problem of possible identification of the two models has been considered above with rather poor accuracy. The analysis of calculations has allowed us to determine that the value η decreases quickly with an increase in σ. Thus, if we assume that $\sigma_\infty = 10$, then η will not exceed a few hundredths and, therefore, the difference in solutions (9) and (14) will constitute, all in all, a few percent. This result holds for both problems with bulk heat evolution and problems with heat supply that is normal to injection.

As calculations show, the one-continuum model has a tendency to "overestimate" the quality of transpiration cooling as compared with the two-continuum model; in particular, for the first boundary-value problems this results in an overestimation of the coolant heating, for the second, it results in underestimating the maximal wall temperature.

We note that this conclusion holds true for problems with heat supply both parallel and normal to the injection.

NOTATION

α, heat-transfer coefficient; T, τ, dimensionless temperature of porous material and coolant, respectively; x, coordinate directed along the direction of motion of the coolant and related to the size of the porous plate in this direction L; y, coordinate normal to the x-axis and related to the size of the porous plate in the direction of yH; $\dot{m}$, coolant mass flow rate per unit cross-sectional area; c_p, coolant heat capacity; λ_x, λ_y, thermal conductivities of the porous plate in corresponding directions; $q^{x(y)}$, x(y) projection of the ambient heat flow; Θ, ambient temperature; T, temperature scale (used as the denominator of the dimensionless temperature); $q = q_V H^2/(\lambda_y T)$, where q_V is the bulk-heat release density. Criteria: $\pi_x = \dot{m}c_p L/\lambda_x$; $\sigma = \alpha_V L/(\dot{m}c_p)$; σ_∞ is the limiting value of σ; $\sigma_{a(b)} = \alpha_{a(b)}/(\dot{m}c_p)$; $\beta_b = \alpha_b L/\lambda_x$; $\beta_{v(n)} = \alpha_{v(n)} H/\lambda_y$; $\varkappa = \lambda_x H^2/(\lambda_y L^2)$; $\pi = \dot{m}c_p H^2/(\lambda_y L)$. Indices: V is referred to the volume of the porous material, a, b, v, and n are referred to the different surfaces of the porous plate, respectively: inlet, outlet, upper, lower (along the y axis); -, +, to the region far from the inlet, far from the outlet of the porous plate.

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